THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2022)
Suggested Solution of Homework 1

Solution:

(1) For any $x \geq 1$, we have $\frac{1}{x} \leq 1 \leq x$. Then $x \notin S$. Thus for any $x \in S$, $x < 1$, i.e., 1 is an upper bound of S.

For any $0 < u < 1$, for any $u < v < 1$. we have $v < 1 < \frac{1}{v}$. Then $v \in S$. Thus u is not upper bound of S. Since $\frac{1}{2} \in S$, any non-positive number cannot be an upper bound of S. Therefore, any upper bound of S is greater than or equal to 1.

Hence, $\sup S = 1$.

For any $x < -1$, $\frac{1}{x} > -1$. Then, $x < \frac{1}{x}$. Therefore any real number less than 1 belongs to S. Since S is not bounded from below, $\inf S$ does not exist in $\mathbb{R}$.

(2) For any $s \in S$, $-s \geq \inf\{-s : s \in S\}$. Then $s \leq -\inf\{-s : s \in S\}$. Thus $-\inf\{-s : s \in S\}$ is an upper bound of S.

For any upper bound v of S, $-v$ is an lower bound of $\{-s : s \in S\}$. Then $-v \leq \inf\{-s : s \in S\}$. Thus $v \geq -\inf\{-s : s \in S\}$.

Hence, $\sup S = -\inf\{-s : s \in S\}$.

(3) For any $x \in A+B$, there exist $a \in A$ and $b \in B$ such that $x = a+b$. Then $a \leq \sup A$ and $b \leq \sup B$. Thus $x = a + b \leq \sup A + \sup B$. Therefore, $\sup A + \sup B$ is an upper bound of $A + B$.

For any upper bound $u < \sup A+\sup B$, $u-\sup A < \sup B$. Then there exists $b \in B$ such that $u - \sup A < b$, i.e., $u - b < \sup A$. Then there exists $a \in A$ such that $u - b < a$, i.e., $u < a + b$. Thus u is not an upper bound of $A + B$. Therefore, any upper bound of $A + B$ is greater than or equal to $\sup A + \sup B$.

Hence, $\sup A+B = \sup A+\sup B$. Similarly, one can show $\inf A+B = \inf A+\inf B$.

(4) By mathematical induction, one can show $n < 2^n$ for any $n \in \mathbb{N}$. Then $\frac{1}{2^n} < \frac{1}{n}$.

By Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{x}$. Since $x > 0$, $\frac{1}{2^n} < \frac{1}{n} < x$.

(5) We change $m \leq x < m + 1$ to $m - 1 \leq x < m$ in Question 5.

By Archimedean property, there exists $k \in \mathbb{N}$ such that $k > x$, i.e., $S := \{k \in \mathbb{N} : k > x\}$ is non-empty. By Well-ordering Principle, S has a smallest element, say m. Then $m - 1 \notin S$. Since $m \in S$, $x < m + 1$. Since $m - 1 \notin S$, $m - 1 \leq x$.

If $m, n \in \mathbb{N}$ are two elements such that $m \neq n$, $m - 1 \leq x < m$ and $n - 1 \leq x < n$. If $m > n$, then $m \geq n + 1 > x + 1$. But $m \leq x + 1$. Contradiction! When $m < n$, a similar argument gives contradiction.